

Semiclassical genetic algorithm with quantum crossover and mutation operations

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Abstract

In order for finding a good individual for a given fitness function in the context of evolutionary computing, we introduce a novel semiclassical quantum genetic algorithm. It has both of quantum crossover and quantum mutation procedures unlike conventional quantum genetic algorithms. A complexity analysis shows a certain improvement over its classical counterpart.

1 Introduction

Continuous development has been performed on genetic algorithms [1–3]. Along with the development of quantum computing [4, 5], quantum-inspired classical algorithms for evolutionary computing have been developed [6–13] (see also a review [14] and references therein). In addition, classical genetic algorithms to evolve quantum circuits have also been studied by several authors [15–29] (see also review articles [30, 31]). These algorithms are, however, designed to work on classical computers. Quantum genetic algorithms (QGAs), in its literal meaning, nonetheless, have gathered comparably little attention and a few works [32–36] have been performed so far. Evolutionary computing on quantum architectures will achieve more attention if there is a scenario to establish significant improvement over classical counterparts. Indeed, Malossini *et al.* [34, 36] claimed that computational complexity of an evolutionary step between generations is exponentially fast in their quantum algorithm in comparison to a classical one. It seems, however, that they overlooked the complexity of index-to-string conversion circuits or, otherwise, overlooked the circuit complexity of a variant of the inversion-about-average operation. Their claim is thus misleading as we will discuss in Sec. 2. Recently, Johannsen *et al.* [37] applied quantum search algorithms to several optimization problems in a certain context of evolutionary computing. Nevertheless, it is obscure how much cost is spent for the internal quantum circuit of a variant of the inversion-about-average operation used for amplitude amplification (namely, the operation denoted as $\mathcal{AS}_0^\phi \mathcal{A}^{-1}$ in the convention of Ref. [38]) in their approach.

Let us briefly summarize the conventional approach of QGAs and its problem. Here, we omit discussions on incomplete or physically unfeasible works on QGAs, which were summarized by Sofge in Ref. [39]. The aim of genetic algorithms is, in most cases, to find an individual (typically, an input string) with a very high fitness value for a given problem. It starts with $O(|A|^w)$ initial individuals where $|A|$ is the size of the alphabet A and w is the length of a schema expected to be a building block for a given problem. Let us restrict the problem by representing each individual as a chromosome encoded as a binary string with the length n . We need $\sim N = 2^n$ initial chromosomes for the worst case. With a quantum register, one may use n qubits to make a superposition of N chromosomes. We regard the probability to find a chromosome on the computational basis as its (normalized) population.

First we briefly overview the selection strategies in conventional QGAs. In short, a selection is an operation to enhance populations of individuals with high fitness values and to decrease those of low fitness values. In quantum computing, it is natural to utilize variants of Grover's algorithm [40] for this purpose. In fact, a variant of the Grover search for *a priori* unknown number of solutions (the Grover-BBHT search) [41] and that for finding the maximum [42, 43] are the essential parts of the QGAs developed in Refs. [33–36]. The query complexity $O(\sqrt{N/r})$ for a variant of the Grover search dominates the total complexity in the QGA of Udrescu *et al.* [33, 35]. For the QGA of Malossini *et al.* [34, 36], there is a different factor to consume time. As mentioned, this will be discussed in detail in Sec. 2.

Second we overview the strategies for crossovers and mutations in conventional QGAs. In a short explanation of these terms, a crossover is to exchange substrings of two chromosomes and a mutation is to flip certain bits of a

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chromosome. These are effective operations to enlarge the search space. So far, quantum crossover operations have not been developed. Malossini *et al.* [34, 36] used classical crossover and mutation operations; Udrescu *et al.* [33, 35] did not use crossover and mutation operations. Johannsen *et al.* [37] introduced quantum mutation operations in an application of quantum amplitude amplification [38] to some optimization problems; nevertheless, crossovers were not used. It is, in fact, in general difficult to manipulate populations of quantum states for handling crossovers if we imitate the classical way in a straightforward manner (See also Ref. [30] which explained the difficulty in a slightly different manner). Picking-up two particular individuals and making a crossover costs $O(n)$ quantum gates if we use $O(n)$ ancillary qubits. Since there are many possible pairs for a crossover at each generation, there is no speedup over a classical crossover. More specifically speaking, this approach needs to look up classical data of chromosomes to specify a pair of chromosomes. Thus there cannot be any speedup. In addition, this requires an exponentially large classical memory in comparison to the size of a quantum register. We will show a different approach to handle crossovers in our algorithm in Sec. 3.

As another direction for developing quantum evolutionary computing, which we do not pursue in this contribution, one may use so-called quantum fixed-point search algorithms [44, 45]. They increase the population of a superposition of target chromosomes by iterative applications of unitary amplitude-amplification operations. Unlike the original Grover search, there is no drawback in an excessive iteration; the probability of finding a target grows monotonically as $1 - \epsilon^{2t+1}$ where ϵ is the probability of finding a non-target chromosome and t is the number of queries. The query complexity of this approach is, however, as large as that of the exhaustive search: When the number r of targets is very small in comparison to N , we have $\epsilon^{2t+1} = (1 - r/N)^{2t+1} \simeq 1 - (2t+1)r/N$, which implies that $O(N/r)$ queries are required to achieve a sufficiently small error probability, say $1/2$. Thus for $r \ll N$, Grover's algorithm should be chosen instead of fixed-point quantum search algorithms. In addition, the asymptotic optimality of ϵ^{2t+1} as an error reduction speed was proved [46] for fixed-point quantum search methods. It is also known that the query complexity of the Grover search is optimal [41] in general as a unitary process for unsorted search. Therefore, it is unlikely¹ that a fixed-point quantum search is effectively used in an evolutionary computing instead of the standard Grover search and its variants.

In this contribution, we propose a semiclassical quantum genetic algorithm that involves a quantum crossover and a quantum mutation. It uses the quantum search for finding the maximum [42, 43] for the selection procedure. Although this selection strategy looks similar to those of conventional QGAs [33–36], it is different in the point that we use tailored inversion operations for the quantum search. Our algorithm achieves a quadratic speedup over its classical counterpart as an algorithm involving all possible crossovers for a chosen crossing site.

The paper is organized in the following way. Section 2 discusses on the algorithm of Malossini *et al.* [34, 36]. Section 3 describes our algorithm. The procedures of the crossover, the mutation, and the selection are developed in the standard quantum circuit model. Computational complexities of the algorithm are evaluated in Sec. 4. The obtained results are discussed in Sec. 5 and summarized in Section 6.

2 A conventional quantum genetic algorithm

Here we discuss the conventional quantum genetic algorithm proposed by Malossini *et al.* [34, 36]. There is another conventional algorithm proposed by Udrescu *et al.* [33, 35], which is not introduced here. The main difference is that Udrescu *et al.* did not employ crossovers and mutations while Malossini *et al.* employed them as classical operations.

The problem we consider is given as follows.

Problem: Suppose there are N individuals with indices $0, \dots, N-1$. There is a fitness function $f : \{0, \dots, N-1\} \rightarrow [0, 1]$. Find one of the individuals with sufficiently large fitness values.

The algorithm of Malossini *et al.* is described as follows.

Algorithm 1 (Malossini *et al.* [34, 36])

Main Procedure:

Start from a generation consisting of $\tilde{N} \leq N$ individuals.

Repeat:

1. Use the *Quantum Selection Procedure* to get one index. Do the same procedure to get another.
2. Classically make a crossover for the two individuals corresponding to the indices obtained in 1. Classically make a proper mutation.
3. Classically replace randomly chosen two individuals with the two offsprings obtained in 2.

¹ There is another drawback in a quantum fixed-point search: it requires a nonstandard oracle operation represented by the unitary gate $\tilde{U}_o = 1 - 2|t\rangle\langle t|$ with $|t\rangle$ a superposition of target states. A standard oracle is represented by the unitary gate $U_o = 1 - 2\sum_m |\tau_m\rangle\langle\tau_m|$ with $|\tau_m\rangle$ the target states. There is no known way to construct \tilde{U}_o from U_o efficiently when we have multiple targets that are unknown in advance. This difficulty exists not merely in the fixed-point search but also in a variant of Grover search using \tilde{U}_o [47].

Quantum Selection Procedure:

- (i) Choose randomly an index y among \tilde{N} possible ones in the present generation and compute its fitness $f(y)$. Set the threshold $F_y \leftarrow f(y)$.
- (ii) Perform τ times:
 - (a) Initialize the quantum registers to $\frac{1}{\sqrt{\tilde{N}}} \sum_x |x\rangle|y\rangle$, where x 's are the indices of the present generation.
 - (b) Apply the Grover-BBHT search [41], where we internally use an oracle that inverts the signs of the marked states, namely states $|x\rangle$ such that $f(x) \geq F_y$. This enhances the amplitudes of marked states after its Grover iteration.
 - (c) Measure the left register in the computational basis and get the new index y' . If $f(y') > F_y$, then $y \leftarrow y'$ and $F_y \leftarrow f(y')$.
- (iii) Return the index y .

They analyzed their algorithm and claimed that the selection procedure internally requires a very small number of oracle calls in contrast to $O(\tilde{N} \log \tilde{N})$ oracle calls required by a classical selection procedure [34, 36]. In short, their claim was that the number of the Grover iterations in step (b) was $O(1)$.

Their complexity analysis was, however, misleading. A more accurate description of the complexity is given in the following way. There are two possible cases we should consider: (A) \tilde{N} is approximately equal to N ; (B) $\tilde{N} \ll N$ holds.

Let us consider the case (A). We can preset the index x to represent the x th individual in a good approximation thanks to $\tilde{N} \simeq N$. Then the given fitness function can be used as it is, without index conversion. This indicates that we use nothing but the standard Grover-BBHT search for N indices. As $\tilde{N} \simeq N$, the quantum selection procedure is called only once or a very small number of times. Suppose there are r nearly-optimal individuals that are acceptable as solutions for a given problem. Inside the quantum selection procedure, as the selection goes on in (ii), the oracle used in step (b) has less marked states. The number of marked states converges to r . Therefore, the query complexity of a single step in a very later selection stage is $O(\sqrt{\tilde{N}/r}) = O(\sqrt{N/r})$. This is the accurate description of the query complexity for the present case and it is same as that Udrescu *et al.* gave for their algorithm [33, 35]. On average, each time of the repetition in (ii) virtually extinguishes a half of the individuals by increasing the threshold. Thus, by considering the sum of the geometric series $\sqrt{N/(2^i r)}$ ($i = 0, 1, 2, \dots$), the average total query complexity is found to be $O(\sqrt{N/r})$.

Let us now consider the case (B). In this case, the quantum selection procedure is called several (or more) times. The important fact is that the chromosomes of individuals in the generation are not consecutive binary strings. The generation contains \tilde{N} binary strings out of N possible ones. There are two options to handle them: (B-1) consecutive integers are assigned as pointers to nonconsecutive chromosomes; (B-2) nonconsecutive chromosomes are used as they are.

(B-1): In this case, the index x is a pointer to the chromosome κ_x for which the fitness function works appropriately. (In short, this is a workaround for the fact that the original Grover-BBHT search should start from a uniform superposition of consecutive indices while the generation consists of nonconsecutive chromosomes.) Therefore, for the superposition $|\psi_0\rangle = \frac{1}{\sqrt{\tilde{N}}} \sum_x |x\rangle|y\rangle$, the fitness function cannot be applied directly. We need a conversion circuit to interpret x as κ_x . Then the fitness function is applied and it returns $f(\kappa_x)$. Each conversion should involve $O(\log N)$ CCNOT gates. Thus, the total circuit depth for the index conversions is $O(\tilde{N} \log N)$. As a result, a single oracle call accompanies the additional time complexity $O(\tilde{N} \log N)$. Indeed, the query complexity is small if \tilde{N} is much smaller than N . However, the cost of index conversions hinders speedup over a classical selection procedure.

(B-2): In this case, the state $|x\rangle$ in the superposition $|\psi_0\rangle = \frac{1}{\sqrt{\tilde{N}}} \sum_x |x\rangle|y\rangle$ keeps the actual chromosome x as an index. One should use the Grover-search routine somehow without index conversions by choosing $|\psi_0\rangle$ as its initial state. Indeed, this is possible [38] in the case where one can provide the operation $L = I - 2|\psi_0\rangle\langle\psi_0|$ as the inversion-about-average operation instead of the standard one for the original Grover search. Nevertheless, L should be generated by sandwiching $I - 2|0\rangle\langle 0|$ with $U_{(a)}$ and $U_{(a)}^\dagger$ in the present context, where the unitary operation $U_{(a)}$ corresponds to the initialization step (a). Thus the circuit complexity of L is $O(\tilde{N} \log N)$ (see, *e.g.*, Ref. [48]). Such an expensive circuit is used subsequent to every query. Obviously, the time complexity in this case is as large as the one in case (B-1).

One may also think of replacing the Grover-search routine with the generalized Grover search for nonconsecutive integer sets [49]. Nevertheless, it is required to find an appropriate unitary transformation replacing the standard Hadamard transformation $H \otimes \dots \otimes H$. There is no known way to find it efficiently when a random integer set is given and target integers are unknown in advance.

There is, in fact, a way to reduce the circuit complexity of L introduced in (B-2) if the algorithm is modified so that it uses an efficient pseudo-randomizer (or, pseudo-scrambler) circuit instead of directly using a random number generator. Our algorithm introduced in the next section takes this approach.

In addition to the above discussions, we should mention that Algorithm 1 has another problem: it takes $O(\tilde{N} \log N)$ space to keep the generation in a classical memory. This is usually quite larger than the $O(\log N)$ space that is enough for quantum search (neglecting the space internally used by an oracle circuit). Our algorithm is designed not to face this problem.

3 Algorithm with quantum crossover and quantum mutation

Conventional quantum genetic algorithms have not achieved improvement in the total complexity over the standard quantum search, and quantum crossover operations have not been developed so far, as we have seen in the previous sections. Here, we show an algorithm using a quantum crossover and a quantum mutation. We also use a variant of the Grover search for the selection procedure, whose query complexity is the main factor of the total time complexity. We have still a certain speedup over a classical counterpart as a genetic algorithm using a crossover. It is novel in the sense that a simultaneous crossover using a superposition is achieved. The speedup partly relies on an efficient internal structure of a pseudo-randomizer circuit, which will be explained in Appendix.

3.1 Algorithm flow

We introduce our algorithm as follows. Here, the quantum register is accessible from subroutines as a kind of global variables.

Algorithm 2

Consider a threshold f_{th} to the fitness values, which is considered to be sufficiently large.

$t \leftarrow 0$.

REPEAT 1.-9.:

1. Construct a pseudo-randomizer R that maps a c -bit string to a pseudo-random n -bit string. It should be implemented as a circuit whose input and output are integer couples $(a, 0)$ and $(a, R(a))$, respectively, where a is a c -bit integer. The circuit should consist of $\text{poly}(cn)$ elementary reversible logic gates. We require R to regard $0_0 \cdots 0_{c-1}$ as an exception and map it to $0_0 \cdots 0_{n-1}$. We also require $\tilde{N} = 2^c \lesssim N = 2^n$. Once constructed, R is fixed until next t . An explicit example to construct R is given in Appendix.
2. **IF** $t = 0$ **THEN** generate a random c -bit string γ and set $z \leftarrow R(\gamma)$ (otherwise, z is the best chromosome found in the $(t - 1)$ th trial) **ENDIF**
3. Generate a random c -bit string γ' and set $u \leftarrow R(\gamma')$.
4. **CALL** `init_reg`(R, z) defined in Sec. 3.2 twice to make two identical quantum states that are both $|\varphi\rangle = \frac{1}{\sqrt{\tilde{N}}} \sum_{x \in X} |a_x\rangle_a |x\rangle$ where a_x is the address pointing to $x \in X$; $X = \{R(q)\}_q \cup \{z\}$ with $q = 0_0 \cdots 0_{c-2} 1_{c-1}, \dots, 1_0 \cdots 1_{c-1}$; subscript “a” stands for the address portion. We write the entire unitary operation of this procedure as U_{init} . That is, $|\varphi\rangle^{\otimes 2} = U_{\text{init}}(|0\rangle_a |0\rangle)^{\otimes 2}$.
5. **CALL** `quantum_crossover`(l) defined in Sec. 3.3 for the current quantum register, which is a 1-point crossover with the crossing site placed between the $(l - 1)$ th and the l th bits for each chromosome. This procedure is an identity map as a quantum operation acting on $|\varphi\rangle^{\otimes 2}$.
6. Apply the quantum mutation (Sec. 3.4) to the current quantum register. We write the entire unitary operation of this procedure as U_{mut} . Apply the same mutation classically to u .
7. **CALL** `quantum_selection`($U_{\text{init}}, U_{\text{mut}}, u$) defined in Sec. 3.5 for the current quantum register and obtain the output chromosome z .
8. **IF** $f(z) \geq f_{\text{th}}$ **THEN RETURN** z and **EXIT** **ENDIF**
9. $t \leftarrow t + 1$. Refresh the quantum register.

3.2 Preparing the initial state of a quantum register

In this section, we define the procedure `init_reg`(R, z).

This procedure is intended to prepare a superposition corresponding to the generation given as a set X of n -bit integers x with $\#X = \tilde{N} = 2^c$. The procedure starts with the quantum state $|0\rangle_a^{\otimes c} |0\rangle^{\otimes n}$. The desired superposition is $|\varphi\rangle = \frac{1}{\sqrt{\tilde{N}}} \sum_{x \in X} |a_x\rangle_a |x\rangle$ with a_x the address pointing x . We opt to use consecutive addresses $0, \dots, 2^c - 1$. The order of time complexity of the procedure equals to the internal circuit complexity $\text{poly}(cn)$ of the pseudo-randomizer R . An explicit example to construct R as a quantum circuit is given in Appendix.

The procedure is defined as follows.

PROCEDURE `init_reg(X)`:

- (i) make a superposition $\frac{1}{\sqrt{\#X}} \sum_{j=0}^{2^c-1} |j\rangle_a |0 \cdots 0\rangle$ by applying $H^{\otimes c} \otimes I^{\otimes n}$ to $|0\rangle_a^{\otimes c} |0\rangle^{\otimes n}$.
- (ii) Apply the pseudo-randomizer R implemented as a unitary operation mapping $|j\rangle_a |0\rangle$ to $|j\rangle_a |x_j\rangle$ with x_j an n -bit pseudo-random number for $j = 1, \dots, 2^c - 1$.
By assumption, $|0\rangle_a |0\rangle$ is mapped to $|0\rangle_a |0\rangle$.
- (iii) Apply a $0_0 \cdots 0_{c-1}$ -controlled $X^{z_0} \otimes \cdots \otimes X^{z_{n-1}}$ to map $|0\rangle_a |0\rangle$ to $|0\rangle_a |z\rangle$, where z_k is the k th bit of z ($k = 0, \dots, n-1$).
- (iv) **RETURN** the current state, namely $|\varphi\rangle$.

3.3 Crossover operation

Here, we construct a quantum crossover procedure `quantum_crossover(l)`. It is a 1-point crossover acting on the chromosomes simultaneously.

Recall that the original generation is given as a set X of n -bit integers x with $\#X = \tilde{N}$. Each x has its left side x^{left} and its right side x^{right} separated by the crossing site. The crossing site is placed between the $(l-1)$ th and the l th qubits as specified by the parameter. Hence, the bit length of x^{left} is l . With the procedure, we generate a superposition of all the children that are combinations of x^{left} 's and x^{right} 's together with their parents with the same weight as children.

The state of the quantum register in the beginning of this procedure is

$$|\varphi\rangle^{\otimes 2} = \left(\frac{1}{\sqrt{\#X}} \sum_{x \in X} |a_x\rangle_a |x^{\text{left}}\rangle |x^{\text{right}}\rangle \right) \otimes \left(\frac{1}{\sqrt{\#X}} \sum_{x' \in X} |a_{x'}\rangle_a |x'^{\text{left}}\rangle |x'^{\text{right}}\rangle \right).$$

This state has the components $|x^{\text{left}}\rangle |x^{\text{right}}\rangle |x'^{\text{left}}\rangle |x'^{\text{right}}\rangle$ besides the addresses. We relabel the qubits so that the middle portion $|x^{\text{right}}\rangle |x'^{\text{left}}\rangle$ is put aside from our minds. Let us conceal them by denoting as $|*_{xx'}\rangle$. In addition, We denote the main portion $|x^{\text{left}} x'^{\text{right}}\rangle$ with the subscript “main”. The state $|\varphi\rangle^{\otimes 2}$ with the new qubit labels is written as

$$\frac{1}{\#X} \sum_{x \in X} \sum_{x' \in X} |a_x a_{x'}\rangle_a |x^{\text{left}} x'^{\text{right}}\rangle_{\text{main}} |*_{xx'}\rangle.$$

We have at most $(\#X)^2$ distinct chromosomes in this state. In this way, all possible crossovers are performed at once by the relabelling. The resultant state is a superposition of all of the children together with their parents. (The parents are involved because the values of x and x' may coincide.) This is desirable as a crossover because sometimes some parents have higher fitness values than any child.

The procedure described above is formally written in the following way.

PROCEDURE `quantum_crossover(l)`:

- (i) We have the quantum register in the state $|\varphi\rangle^{\otimes 2}$. The original labels of its qubits are $0, \dots, 2c + 2n - 1$. We relabel them as $\underbrace{0, \dots, c-1}_c, \underbrace{2c, \dots, 2c+l-1}_l, \underbrace{2c+n, \dots, 2c+2n-l-1}_{n-l}, \underbrace{c, \dots, 2c-1}_c, \underbrace{2c+2n-l, \dots, 2c+2n-1}_l, \underbrace{2c+l, \dots, 2c+n-1}_{n-l}$.
- (ii) **RETURN**

As is obvious, this procedure is an identity map as a quantum operation. Once the crossover process is completed, one may use a mutation as an option. This is going to be explained in the next section.

3.4 Mutation

In classical genetic algorithms, randomly selected chromosomes are affected by a mutation, such as bit flips acting on randomly-chosen places. Here, we consider the mutation described below. Although it is written as a classical routine, it can be trivially interpreted to a quantum circuit.

PROCEDURE `tmp_mut`:

- (i) Let us randomly generate the first template like `***0*1***1**0****1*`, which specifies a schema to mutate. Using this template, we pick up chromosomes with specified bits like 0,1,1,0,1 in the specified places.

- (ii) We also use the second template like ***X*****X**** in which X's can be placed only on the places where *'s (namely, "don't care" symbols) are placed in the first template. Using this template, we apply the bit flip X to the specified places of the chromosomes picked up in (i).
- (iii) **RETURN**

Note: Technically, we often wish to avoid a mutation for the best chromosome z found so far by the present time step. This is realized by choosing the first template so that this does not happen.

As a quantum circuit, **tmp_mut** is realized by a multiple-bit controlled multiple-bit NOT gate. In the present example, the gate is "**0-controlled 1-controlled 1-controlled 0-controlled 1-controlled NOT NOT**" with control bits specified by the first template and the target bits specified by the second template. The gate acts on the portion $|x^{\text{left}}x'^{\text{right}}\rangle_{\text{main}}$; the addresses a_x and $a_{x'}$ are untouched. Therefore, the resultant state can be written as

$$|\tilde{\varphi}\rangle = \frac{1}{\#X} \sum_{x \in X} \sum_{x' \in X} |a_x a_{x'}\rangle_a |\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}}\rangle_{\text{main}} |*_{xx'}\rangle, \quad (1)$$

where $(\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}})$'s are the chromosomes after the mutation process.

The next step is to apply a natural selection to the chromosomes living in the superposition $|\tilde{\varphi}\rangle$.

3.5 Selection

In this section, we introduce our selection procedure. It is intended to find a chromosome having the maximum fitness among $(\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}})$'s. It utilizes the quantum search for finding the maximum [42, 43]. As we have mentioned, conventional QGAs [33–36] have similar selection strategies. The difference from them is that we use tailored inversion operations for the quantum search.

This procedure is called with three arguments: U_{init} , U_{mut} , and u . We have the state (1) at the beginning of this procedure. The procedure is formally defined as below. Here, $k_{\text{term}} = \eta \times \lceil (45/2)\tilde{N} + (28/5)(\log_2 \tilde{N})^2 \rceil$ with integer constant $\eta \geq 1$.

PROCEDURE `quantum_selection`(U_{init} , U_{mut} , u):

FOR $k \leftarrow 0$ **TO** $k_{\text{term}} - 1$:

- (i) Set

$$U_1 = I \otimes I \otimes I - 2 \sum_{f(y) \geq f(u)} I \otimes |y\rangle\langle y| \otimes I$$

where the left and the right I 's act on the address states $|a_x a_{x'}\rangle$ and the states $|*_{xx'}\rangle$, respectively. This is the oracle function that can be implemented as follows. First, we attach ancillary qubits as blocks (I) and (II) in the state $|0 \cdots 0\rangle_{\text{(I)}} |-\rangle_{\text{(II)}}$ with $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. For each xx' , $f(\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}})$ is set as a value of the block (I) by using f implemented as a quantum circuit. Let us write this operation as U_f . Then we compare the block (I) with $f(u)$; we flip the qubit (II) if $f(\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}}) \geq f(u)$. We apply U_f^\dagger (this disentangles the ancillary qubits from the main register) and remove the ancillary qubits.

- (ii) Set

$$U_2 = I \otimes I \otimes I - 2|\tilde{\varphi}\rangle\langle\tilde{\varphi}|.$$

This operation is composed in the following way.

$$U_2 = U_{\text{mut}} U_{\text{init}} [I - 2(|0\rangle^{\otimes c} |0\rangle^{\otimes n} \langle 0|^{\otimes c} \langle 0|^{\otimes n})^{\otimes 2}] U_{\text{init}}^\dagger U_{\text{mut}}^\dagger,$$

where we also relabel qubits according to `quantum_crossover`(l).

- (iii) Apply the Grover-BBHT search [41] for which we use U_1 and U_2 instead of the standard operations, namely, the inversion operation for targets and the inversion-about-average operation, respectively. As for the starting state of the search, we use $|\tilde{\varphi}\rangle$.
- (iv) Measure the main register and obtain the chromosome u' .
IF $f(u') > f(u)$ **THEN** $u \leftarrow u'$ **ENDIF**

END FOR

RETURN u .

Let us explain on the number of iterations, k_{term} . The above procedure is same as the well-known quantum search algorithm for finding the maximum [42, 43] except for the definitions of the inversion operation for targets and the inversion-about-average operation. In other words, we perform the quantum search for finding the maximum in the subspace $\text{span}\{|a_x a_{x'}\rangle_a |\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}}\rangle_{\text{main}} |*_{xx'}\rangle\}$. As proved by Dürr and Høyer [42], the probability for the output to be the maximum is at least $1/2$ if the number of iterations is $\lceil (45/2)\sqrt{M} + (7/5)(\log_2 M)^2 \rceil$ with M the number of indices. In the present context, we have $M = \tilde{N}^2$ since there are \tilde{N}^2 distinct addresses. Therefore, after k_{term} iterations, we find the output chromosome having the maximum fitness among $(\tilde{x}^{\text{left}} \tilde{x}'^{\text{right}})$'s, with the probability at least $1 - (1/2)^\eta$. By setting η sufficiently large, say, around 16 to 24, we have the desired output with almost certainty.

4 Computational cost

We evaluate the computational cost of each process of Algorithm 2 in Sec. 4.1. The total computational cost to handle a generation is derived in Sec. 4.2 and compared with that of classical counterpart in Sec. 4.3.

4.1 Costs of each procedure

For a single call of each procedure, the costs are evaluated as follows.

Cost of the initialization The procedure to prepare the initial state, described in Sec. 3.2, uses $\text{poly}(cn) = \text{poly}(\log \tilde{N} \log N)$ elementary quantum gates.

Cost of the crossover The crossover described in Sec. 3.3 makes use of two identical $(c + n)$ -qubit states. This procedure does not use any quantum operation but relabels qubits. This takes $O(\log N)$ time.

Cost of the mutation The mutation described in Sec. 3.4 involves a single gate that looks like, say, $* - \text{NOT} - * - C_0 - * - C_1 - * - * - * - C_1 - * - * - C_0 - * - * - \text{NOT} - * - C_1 - *$ with multiple control bits (0-control C_0 's and 1-control C_1 's) and multiple NOT gates placed according to the corresponding templates, where symbol $*$ stands for an untouched qubit. Such a gate can be realized by $O(\log N)$ elementary quantum gates with $O(\log N)$ ancillary qubits.

Cost of the natural selection The natural selection described in Sec. 3.5 consumes $O(\tilde{N})$ queries to find the best chromosome among the set of at most \tilde{N}^2 chromosomes. Each query accompanies the operation U_2 that invokes U_{init} and U_{mut} , and also their inverse operations. It is easy to find that the internal cost of U_2 is $\text{poly}(cn)$ according to the costs for U_{init} and U_{mut} . In addition, the cost to prepare the starting state is $\text{poly}(cn)$. We may also mention that, the internal cost of the fitness function is a certain small factor, typically $\text{poly}(n)$, as conventionally assumed [2]. Therefore, the circuit complexity² of the procedure is $O(\tilde{N} \text{poly}(\log \tilde{N} \log N))$. As for space, we use $\text{poly}(\log N)$ qubits in total, considering a typical fitness function consuming $\text{poly}(n)$ space. When the fitness function is designed to use $O(n)$ space, $O(\log N)$ qubits are enough, although we do not assume this case for evaluating the space complexity.

4.2 Total cost

Comparing the costs of the four procedures, the dominant cost is the circuit complexity for the selection procedure. Therefore, we find that our algorithm uses

$$O(\tilde{N} \text{poly}(\log \tilde{N} \log N)) \quad (2)$$

elementary quantum gates for each t . This is the time complexity of our algorithm for handling each generation. As for the space complexity, we spend $\text{poly}(\log N)$ qubits and $O(\log N)$ classical bits as is easily evaluated from the description of Algorithm 2.

4.3 Comparison with a classical counterpart

The classical counterpart of our algorithm is the following algorithm. As is obvious from its structure, it should have the same output and the same number of iterations as our algorithm with almost certainty as long as the same random seed is used for step 1 to construct R for each value of t .

Algorithm 3

²We assume that the circuit depth is on the order of the circuit size, namely, the number of elementary quantum gates.

Consider a threshold f_{th} to fitness values, which is considered to be sufficiently large.

$t \leftarrow 0$.

REPEAT 1.-7.:

1. Construct the pseudo-randomizer R found in Algorithm 2 as a classical function. Fix R until next t . Using R , we generate a generation $X = \{R(q)\}$ with $q = 0_0 \cdots 0_{c-2} 1_{c-1}, \dots, 1_0 \cdots 1_{c-1}$. We require $\tilde{N} = 2^c \not\approx N = 2^n$.
(Thus, we have $\tilde{N} - 1$ n -bit-length chromosomes in X presently.)
2. **IF** $t \neq 0$ **THEN** put z into X (z is the best chromosome found in the $(t - 1)$ th trial) **ELSE** generate a random c -bit string γ and put $R(\gamma)$ into X **ENDIF**
(Now we have \tilde{N} chromosomes in X .)
3. Split all $x \in X$ between the $(l - 1)$ th bit and the l th bit. This makes l -bit strings x^{left} 's and $(n - l)$ -bit strings x^{right} 's. Generate the set \hat{X} consisting of all x 's and all of their children that are all the combinations of x^{left} 's and x^{right} 's.
4. Apply the template-based mutation **tmp_mut**, introduced in Sec. 3.4, as a classical procedure to all chromosomes in \hat{X} .
5. Find the chromosome z having the best fitness value among those in \hat{X} .
6. **IF** $f(z) \geq f_{\text{th}}$ **THEN RETURN** z and **EXIT** **ENDIF**
7. $t \leftarrow t + 1$.

The computational cost of each procedure of Algorithm 3 is as follows.

- Generating X in the steps 1.-2. takes $O(\tilde{N} \text{poly}(cn))$ basic operations.
- We need to use $O(n\tilde{N}^2)$ space and $O(n\tilde{N}^2)$ basic operations to perform all the crossovers among \tilde{N} parents.
- Mutations acting on the individuals of \hat{X} takes $O(n\tilde{N}^2)$ basic operations.
- The selection to find the best individual from \hat{X} takes $O(\tilde{N}^2 \text{poly}(n))$ basic operations considering the cost $\text{poly}(n)$ of calculating a fitness value.

As $n = \log_2 N$ and $c = \log_2 \tilde{N}$, the time and space complexities are $O(\tilde{N}^2 \text{poly}(\log N))$ and $O(\tilde{N}^2 \log N)$, respectively, for each t , i.e., for handling each generation.

In contrast, as we have seen in Sec. 4.2, the time and space complexities of our semiclassical quantum genetic algorithm (Algorithm 2) are $O(\tilde{N} \text{poly}(\log \tilde{N} \log N))$ and $\text{poly}(\log N)$, respectively, for each t . Therefore, neglecting the difference between $\text{poly}(\log \tilde{N} \log N)$ and $\text{poly}(\log N)$, we have achieved a quadratic speedup over its classical counterpart together with an exponential reduction in space.

The classical counterpart has been constructed by keeping the one-by-one correspondence with the quantum algorithm. Thus there is a possibility that a better classical algorithm with the same behavior exists. This is in fact the case for Algorithm 3. For a fairer comparison, now we reform Algorithm 3 and reduce its space complexity. The following algorithm has the same output and the same number of repetitions as Algorithm 3 while its space complexity is exponentially reduced. We use the same pseudo-randomizer construction and the same mutation procedure as before. Of course, we set $\tilde{N} = 2^c$ and $N = 2^n$ for integers c and n satisfying $1 \leq c < n$.

Algorithm 4

Consider a threshold f_{th} to fitness values, which is considered to be sufficiently large.

$t \leftarrow 0$.

REPEAT 1.-8.:

1. Construct the pseudo-randomizer $R : \{0, 1\}^c \rightarrow \{0, 1\}^n$ as a classical operation.
2. Construct the mutation process as a map M as a classical operation.
3. **IF** $t = 0$ **THEN** for a random $\gamma \in \{1, \dots, \tilde{N} - 1\}$, $z \leftarrow R(\gamma)$ (otherwise, z is the best chromosome found in the $(t - 1)$ th step) **ENDIF**
4. For a random $\gamma' \in \{1, \dots, \tilde{N} - 1\}$, $j \leftarrow R(\gamma')$.
5. **FOR** $a \leftarrow 0$ **TO** $\tilde{N} - 1$:
IF $a = 0$ **THEN** $x \leftarrow z$ **ELSE** $x \leftarrow R(a)$ **ENDIF**
FOR $b \leftarrow 0$ **TO** $\tilde{N} - 1$:
IF $b = 0$ **THEN** $y \leftarrow z$ **ELSE** $y \leftarrow R(b)$ **ENDIF**
Crossover x and y and obtain children v and w .
Find the best chromosome g among the chromosomes $M(x)$, $M(y)$, $M(v)$, and $M(w)$.
IF $f(g) > f(j)$ **THEN** $j \leftarrow g$ **ENDIF**
END FOR
END FOR

6. $z \leftarrow j$.
7. **IF** $f(z) \geq f_{\text{th}}$ **THEN RETURN** z and **EXIT** **ENDIF**
8. $t \leftarrow t + 1$.

In this algorithm, R is called $O(\tilde{N}^2)$ times for each value of t . We know that R internally takes $\text{poly}(cn)$ time. Therefore, this algorithm spends $O(\tilde{N}^2 \text{poly}(\log \tilde{N} \log N))$ time for handling each generation. As for the space complexity, it spends only $\text{poly}(\log N)$ space, which is clear from the algorithm structure.

In comparison to this enhanced classical algorithm, our semiclassical quantum algorithm still has a quadratically small time complexity as shown in Eq. (2).

5 Discussion

How to perform crossovers in a quantum manner was a pending problem in conventional quantum genetic algorithms [33–36]. In fact, a selective crossover for specific two chromosomes is expensive when they are component states of a superposition. Even if we attach address states pointing to the component states, we need to look up the classical data of the chromosomes to construct a quantum circuit realizing the unitary operation for this purpose, or more specifically, to place address-controlled bit-flips appropriately. In our algorithm (Algorithm 2), we have avoided to mimic a classical way and chosen a different approach. We use two identical copies of a superposition corresponding to a generation and utilize relabelling of qubits so as to handle all possible combinations of substrings simultaneously. Obviously, the classical counterpart of our algorithm is the one that seeks for the best chromosome (after a mutation) among all possible crossovers for a chosen crossing site for each generation. Comparing our algorithm with the classical counterpart, we concluded that we have achieved a considerable reduction in the computational cost.

One may, however, claim that usually at most several crossovers are performed for a single generation in a classical genetic algorithm. Indeed, our algorithm is not aimed to be a quantum counterpart of a common classical genetic algorithm. As we discussed in Sec. 2, a straight-forward conversion of a common classical algorithm into quantum one by simply incorporating a quantum search into the selection procedure has a problem: we need to either interpret nonconsecutive integers to consecutive ones or use an expensive construction for the inversion-about-average operation, in order to perform the Grover-BBHT search, which causes a significant loss of performance. This problem should be resolved so as to find an appropriate quantum counterpart for the common case. Seemingly, the following work-around looks fine: (i) Use the pseudo-randomizer R used in Algorithm 2 instead of a random number generator to generate initial chromosomes of a generation. (ii) Apply a small number of crossovers. (iii) Use a selection procedure similar to that of Algorithm 2. Nevertheless, as we have discussed, it is not known how to construct the procedure (ii) as a unitary operation without the expensive process of looking up classical data of chromosomes in the generation. Therefore, it is the fact that a meaningful quantum counterpart is not easily found for a common classical genetic algorithm.

In view of the search space covered by each generation, a simultaneous crossover is, of course, desirable. Using a superposition for this purpose was discussed [33, 35] but not developed previously. In this sense, we have made a meaningful improvement by introducing Algorithm 2.

There have not been many studies on quantum genetic algorithms in the literal meaning (i.e., not in the meaning of quantum-inspired classical algorithms) so far. It is hoped that useful quantum extensions other than our proposal will be developed in future.

6 Summary

We have proposed a semiclassical quantum genetic algorithm whose crossover, mutation, and selection procedures have been constructed as quantum routines so that quantum parallelism is effectively used. Our complexity analysis has shown that the algorithm achieves a quadratic speedup over its classical counterpart.

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Appendix: An example of constructing the pseudo-randomizer R

In this appendix, we show an example to construct the pseudo-randomizer R used in Algorithms 2, 3, and 4. It should map a c -bit string to a pseudo-random n -bit string except for $0_0 \cdots 0_{c-1}$ that is mapped to $0_0 \cdots 0_{n-1}$. Its internal

circuit complexity should be $\text{poly}(cn)$. As we use a quantum circuit to realize it in Algorithm 2, it is desirable to employ a circuit structure that is originally unitary.

Consider inputs $a \in \{0,1\}^c$. We design a circuit that maps $a_0 \cdots a_{c-1} 0_0 \cdots 0_{n-1}$ to $a_0 \cdots a_{c-1} \kappa_0 \cdots \kappa_{n-1}$ with $\kappa = R(a)$ an n -bit pseudo-random number (here, $a_0 \cdots a_{c-1}$ and $\kappa_0 \cdots \kappa_{n-1}$ are the binary representations of a and κ). By the definition of R , the circuit preserves $0_0 \cdots 0_{c-1} 0_0 \cdots 0_{n-1}$. This circuit is generated by the following function.

FUNCTION `gen_r_circ()`:

Comment: We use wires $v_0, \dots, v_{c-1}, w_0, \dots, w_{n-1}$.

FOR $i \leftarrow 0$ **TO** $c-1$:

- (1) Use a random number generator to generate an n -bit integer γ . Write its binary representation as $\gamma_0 \cdots \gamma_{n-1}$.
- (2) Using the wire v_i as the control wire (namely, the control bit), output the gate $\text{controlled-}(X_0)^{\gamma_0} \otimes \cdots \otimes (X_{n-1})^{\gamma_{n-1}}$ with X_k the bit flip gate acting on the wire w_k ($k = 0, \dots, n-1$). In this gate, the bit flips are active under the condition that $v_i = 1$.

END FOR

Obviously, the circuit output from this function can be directly used as a quantum circuit. Using the circuit $C = \text{gen_r_circ}()$, we have $|a\rangle|0\rangle \xrightarrow{C} |a\rangle|R(a)\rangle \quad \forall a \in \{0,1\}^c$. The circuit complexity of C is $O(cn)$ because, for each i , at most n CNOT gates are used to decompose the gate output from step (2). In addition, `gen_r_circ()` spends $O(c \text{ poly}(n))$ time when a common random number generator [50, 51] is used in step (1).

Note that the function `gen_r_circ()` is called only once for each t , in the beginning of step 1 in Algorithms 2, 3, and 4. We have only to reuse the circuit C for the use of the pseudo-randomizer until t is incremented.

It is expected that outputs from the circuit C possess good uniformity if we use a good random number generator in step (1) of `gen_r_circ()` for generating C . Let us write γ as $\gamma(i)$ to emphasize its dependence on i . For a nonzero input $a_0 \cdots a_{c-1}$, the k th bit of the output $R(a)$ is $\sum_{i=0}^{c-1} a_i \cdot \gamma_k(i) \bmod 2$. This indicates that, for two different inputs a and a' , the k th bits of $R(a)$ and $R(a')$ differ with probability $1/2$ in the ideal case where $\gamma(i)$'s are generated from a true random number generator. This is because a and a' differ by at least a single bit. It also indicates that two different bits, the k th and the k' th bits, of $R(a)$ for a nonzero input a differ with the probability $1/2$ in the ideal case. This is because $\gamma_k(i)$ and $\gamma_{k'}(i)$ differ with the probability $1/2$.

Now we show the result of our numerical test of C . We tried statistical tests of randomness [50, 52] to test pseudo-random numbers output from C , using NIST's Statistical Test Suite (STS) (version 2.1.1) [52]. We set $c = 10$ and $n = 32$. Mersenne Twister (MT) (version mt19937ar) [51] was used to generate γ in step (1) of `gen_r_circ()`. We used the seed value 121212 and did not reset MT during the circuit generation. The circuit C output from `gen_r_circ()`, of course, consisted of 10 outputs from step (2). For this C , we used the inputs $a \in \{0,1\}^c \setminus \{0_0 \cdots 0_{c-1}\}$ from smaller to larger and obtained corresponding outputs $R(a)$ by numerical computation. We obtained 1023×32 bits in total in the outputs, since $2^{10} - 1 = 1023$. We regarded them as a serial bit string from left to right and used STS in its default setting to test the string. In the execution of STS, we used 25 binary sequences with length 1200 as samples from the string. The following tests were tried with the default parameter values in STS: Frequency Test, Block Frequency Test, Cumulative Sums Test, Runs Test, Longest-Run-of-Ones Test, Binary Matrix Rank Test, Spectral DFT Test, and Serial Test. The string passed the tests except for Binary Matrix Rank Test. It should be noted that the input length was too small for the binary matrix rank test [52]. In addition, randomness is not very strictly required for the use in evolutionary computing. Therefore, considering the tests that the string passed, we may claim that `gen_r_circ()` generates a usable pseudo-randomizer circuit for our algorithm.

We conducted another test: We generated ten circuits by calling `gen_r_circ()` ten times without resetting MT, using the seed value 676767. For each circuit, we performed the same process as above to obtain the serial bit string. We obtained ten serial bit strings in total and tested the concatenated string using STS. As samples input to STS, we used 25 binary sequences with length 12000. The concatenated string passed the tests except for Binary Matrix Rank Test and Spectral DFT Test. It was unexpected that it did not pass the spectral DFT test. It requires a further investigation to reveal the reason of this phenomenon.

The results of the first and the second tests are summarized in Table 1. In summary for this appendix, we found that a pseudo-randomizer circuit whose outputs possess enough randomness can be generated by the function `gen_r_circ()` for the use in our algorithm. It is hoped that the function will be improved so as to achieve better randomness for the sake of general use.

References

- [1] J. H. Holland. *Adaptation in Natural and Artificial Systems: An Introductory Analysis with Applications to Biology, Control and Artificial Intelligence*. The University of Michigan Press, Ann Arbor, MI, 1975.

	1st Test	2nd Test
Frequency Test	0.021262	0.186566
Block Frequency Test	0.105618	0.001156
Cumulative Sums Test (Forward)	0.105618	0.029796
Cumulative Sums Test (Reverse)	0.001691	0.057146
Runs Test	0.141256	0.010606
Longest-Run-of-Ones Test	0.875539	0.875539
Binary Matrix Rank Test	0.000000 *	0.000000 *
Spectral DFT Test	0.000533	0.000000 *
Serial Test	0.041438	0.186566

Table 1: List of test results for the first and the second tests we performed (see the text for the details of the tests). Each value is a P -value (see Ref. [52] for its meaning for each test). The asterisk * indicates a failure to pass the test. We should mention that the values for the Longest-Run-of-Ones Test accidentally coincided, while the internally-used results for sample sequences were different.

- [2] D. E. Goldberg. *Genetic Algorithms in Search, Optimization, and Machine Learning*. Addison-Wesley, Reading, MA, 1989.
- [3] M. Mitchell. *An Introduction to Genetic Algorithms*. MIT Press, Cambridge, MA, 1996.
- [4] J. Gruska. *Quantum Computing*. McGraw-Hill, London, 1999.
- [5] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge, 2000.
- [6] A. Narayanan and M. Moore. Quantum-inspired genetic algorithms. In *Proceedings of the IEEE 3rd International Conference on Evolutionary Computation (ICEC96)*, pages 61–66, Nagoya, Japan, 20–22 May 1996. IEEE Press, Piscataway, NJ, 1996.
- [7] K.-H. Han and J.-H. Kim. Genetic quantum algorithm and its application to combinatorial optimization problem. In Congress on Evolutionary Computation, editor, *Proceedings of the 2000 Congress on Evolutionary Computation (CEC2000)*, pages 1354–1360, La Jolla, CA, 16–19 July 2000. IEEE Press, Piscataway, NJ, 2000.
- [8] K.-H. Han and J.-H. Kim. Quantum-inspired evolutionary algorithm for a class of combinatorial optimization. *IEEE Trans. Evol. Comput.*, 6(6):580–593, 2002.
- [9] K.-H. Han and J.-H. Kim. Quantum-inspired evolutionary algorithms with a new termination criterion, h_e gate, and two-phase scheme. *IEEE Trans. Evol. Comput.*, 8(2):156–169, 2004.
- [10] S. Nakayama, T. Imabeppu, S. Ono, and I. Iimura. Consideration on pair swap strategy in quantum-inspired evolutionary algorithm. *IEICE Trans. Inf. Sys.*, J89-D(9):2134–2139, 2006. in Japanese.
- [11] S. Nakayama, T. Imabeppu, and S. Ono. Pair swap strategy in quantum-inspired evolutionary algorithm, 2006. in the Late-breaking papers of the 2006 Genetic and Evolutionary Computation Conference (GECCO-2006), Seattle, WA, 8–12 July 2006.
- [12] M. Chen and H. Quan. Quantum-inspired evolutionary algorithm based on estimation of distribution. In *Proceedings of the 2nd International Conference on Bio-Inspired Computing: Theories and Applications (BIC-TA 2007)*, pages 17–19, Zhengzhou, China, 14–17 September 2007. IEEE Press, Piscataway, NJ, 2007.
- [13] R. Liao, X. Wang, and Z. Qin. A novel quantum-inspired genetic algorithm with expanded solution space. In J. Lawry *et al.*, editor, *Proceedings of the 2010 Second International Conference on Intelligent Human-Machine Systems and Cybernetics (IHMSC 2010)*, pages 192–195, Nanjing, China, 26–28 August 2010. IEEE Computer Society, Los Alamitos, CA, 2010.
- [14] G. Zhang. Quantum-inspired evolutionary algorithms: a survey and empirical study, 2010. J. Heuristics, online first, 29 June 2010, 49 pages, DOI:10.1007/s10732-010-9136-0.
- [15] C. P. Williams and A. G. Gray. Automated design of quantum circuits. In C. P. Williams, editor, *Quantum Computing and Quantum Communications: First NASA International Conference (LNCS 1509)*, pages 113–125, Palm Springs, CA, 17–20 February 1998. Springer-Verlag, Berlin, 1999.

- [16] L. Spector, H. Barnum, and H.J. Bernstein. Genetic programming for quantum computers. In J. R. Koza *et al.*, editor, *Genetic Programming 1998: Proceedings of the Third Annual Conference (GP-98)*, pages 365–374, Madison, WI, 22-25 July 1998. Morgan Kaufmann, San Francisco, 1998.
- [17] H. Barnum, H. J. Bernstein, and L. Spector. A quantum circuit for OR, 1999. arXiv: quant-ph/9907056.
- [18] L. Spector, H. Barnum, H.J. Bernstein, and N. Swamy. Finding a better-than-classical quantum AND/OR algorithm using genetic programming. In P. J. Angeline *et al.*, editor, *Proceedings of the 1999 Congress on Evolutionary Computation (CEC1999)*, pages 2239–2246, Washington, D.C., 6-9 July 1999. IEEE Press, Piscataway, NJ, 1999.
- [19] T. Yabuki and H. Iba. Genetic algorithms for quantum circuit design—evolving a simpler teleportation circuit. In D. Whitley *et al.*, editor, *Proceedings of the 2000 Genetic and Evolutionary Computation Conference (GECCO-2000)*, pages 425–430, Las Vegas, NV, 8-12 July 2000. Morgan Kaufmann, San Francisco, 2000.
- [20] B. I. P. Rubinstein. Evolving quantum circuits using genetic programming. In Congress on Evolutionary Computation, editor, *Proceedings of the 2001 Congress on Evolutionary Computation (CEC2001)*, pages 144–151, Seoul, Korea, 27-30 May 2001. IEEE Press, Piscataway, NJ, 2001.
- [21] M. Lukac and M. Perkowski. Evolving quantum circuits using genetic algorithm. In A. Stoica *et al.*, editor, *Proceedings of the 2002 NASA/DoD Conference on Evolvable Hardware*, pages 177–181, Alexandria, VA, 15-18 July 2002. IEEE Computer Society, Los Alamitos, CA, 2002.
- [22] A. Leier and W. Banzhaf. Evolving Hogg’s quantum algorithm using linear-tree GP. In Erick Cantú-Paz *et al.*, editor, *Proceedings of the Genetic and Evolutionary Computation Conference 2003 (GECCO-2003), Part I (LNCS 2723)*, pages 390–400, Chicago, IL, 12-16 July 2003. Springer-Verlag, Berlin, 2003.
- [23] L. Spector. *Automatic Quantum Computer Programming: A Genetic Programming Approach*. Springer, New York, 2004, Paperback Ed. 2007.
- [24] A. Leier and W. Banzhaf. Comparison of selection strategies for evolutionary quantum circuit design. In K. Deb *et al.*, editor, *Proceedings of the Genetic and Evolutionary Computation Conference 2004 (GECCO-2004), Part II (LNCS 3103)*, pages 557–568, Seattle, WA, 26-30 June 2004. Springer-Verlag, Berlin, 2004.
- [25] P. Massey, J. A. Clark, and S. Stepney. Evolving quantum circuits and programs through genetic programming. In K. Deb *et al.*, editor, *Proceedings of the Genetic and Evolutionary Computation Conference 2004 (GECCO-2004), Part II (LNCS 3103)*, pages 569–580, Seattle, WA, 26-30 June 2004. Springer-Verlag, Berlin, 2004.
- [26] M. Lukac, M. Perkowski, H. Goi, M. Pivtoraiko, C. H. Yu, K. Chung, H. Jee, B.-G. Kim, and Y.-D. Kim. Evolutionary approach to quantum and reversible circuits synthesis. In S. N. Yanushkevich, editor, *Artificial Intelligence in Logic Design*, pages 201–257. Kluwer Academic Publisher, Dordrecht, 2004.
- [27] P. Massey, J. A. Clark, and S. Stepney. Human-competitive evolution of quantum computing artefacts by genetic programming. *Evol. Comput.*, 14(1):21–40, 2006.
- [28] S. Ding, Z. Jin, and Q. Yang. Evolving quantum oracles with hybrid quantum-inspired evolutionary algorithm, 2006. arXiv: quant-ph/0610105.
- [29] L. Spector and J. Klein. Machine invention of quantum computing circuits by means of genetic programming. *AI EDAM*, 22:275–283, 2008.
- [30] G. A. Giraldis, R. Portugal, and R. N. Thess. Genetic algorithms and quantum computation, 2004. arXiv: cs/0403003.
- [31] A. Gepp and P. Stocks. A review of procedures to evolve quantum algorithms, 2007. arXiv: 0708.3278.
- [32] B. Rylander, T. Soule, J. Foster, and J. Alves-Foss. Quantum evolutionary programming. In L. Spector *et al.*, editor, *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO-2001)*, pages 1005–1011, San Francisco, CA, 7-11 July 2001. Morgan Kaufmann, San Francisco, 2001.
- [33] M. Udrescu, L. Prodan, and M. Vlăduțiu. Grover’s algorithm and the evolutionary approach of quantum computation, 2004. ACSA Report, "Politehnica" University of Timisoara, 15 Oct. 2004, <http://www.acsa.upt.ro/publications/index.htm>.
- [34] A. Malossini, E. Blanzieri, and T. Calarco. QGA: quantum genetic algorithm, 2004. Technical Report: #DIT-04-105, Dec. 2004, Univ. Trento, <http://www.dit.unitn.it>.

- [35] M. Udrescu, L. Prodan, and M. Vlăduțiu. Implementing quantum genetic algorithms: A solution based on Grover's algorithm. In M. Alderighi *et al.*, editor, *Proceedings of the 3rd Conference on Computing Frontiers*, pages 71–81, Ischia, Italy, 3-5 May 2006. ACM Press, New York, 2006.
- [36] A. Malossini, E. Blanzieri, and T. Calarco. Quantum genetic optimization. *IEEE Trans. Evol. Comput.*, 12(2):231–241, 2008.
- [37] D. Johannsen, P. P. Kuru, and J. Lengler. Can quantum search accelerate evolutionary algorithms? In J. Branke *et al.*, editor, *Proceedings of the 12th Annual Genetic and Evolutionary Computation Conference (GECCO-2010)*, pages 1433–1440, Portland, OR, 7-11 July 2010. ACM, New York, NY, 2010.
- [38] G. Brassard, P. Høyer, and A. Tapp. Quantum counting. In K. G. Larsen, S. Skyum, and G. Winskel, editors, *Proceedings of Automata, Languages and Programming, 25th International Colloquium (ICALP'98)* (LNCS 1443), pages 820–831, Aalborg, Denmark, 13-17 July 1998. Springer-Verlag, Berlin, 1998. arXiv: quant-ph/9805082.
- [39] D. A. Sofge. Prospective algorithms for quantum evolutionary computation, 2008. arXiv: 0804.1133.
- [40] L. K. Grover. A fast quantum mechanical algorithm for database search. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing (STOC 1996)*, pages 212–219, Philadelphia, PA, 22-24 May 1996. ACM Press, New York, 1996.
- [41] M. Boyer, G. Brassard, P. Høyer, and A. Tapp. Tight bounds on quantum searching. *Fortschr. Phys.*, 46:493–505, 1998.
- [42] C. Dürr and P. Høyer. A quantum algorithm for finding the minimum, 1996. arXiv: quant-ph/9607014.
- [43] A. Ahuja and S. Kapoor. A quantum algorithm for finding the maximum, 1999. arXiv: quant-ph/9911082.
- [44] L. K. Grover. Fixed-point quantum search. *Phys. Rev. Lett.*, 95:150501–1–4, 2005.
- [45] T. Tuli, L. K. Grover, and A. Patel. A new algorithm for fixed point quantum search. *Quant. Inf. Comput.*, 6:483–494, 2006.
- [46] S. Chakraborty, J. Radhakrishnan, and N. Raghunathan. Bounds for error reduction with few quantum queries. In C. Chekuri, K. Jansen, J.D.P. Rolim, and L. Trevisan, editors, *Proceedings of the 9th International Workshop on Randomization and Computation (RANDOM 2005)* (LNCS 3624), pages 245–256, Berkeley, CA, 22-24 August 2005. Springer-Verlag, Berlin, 2005.
- [47] Y. Tanaka, T. Ichikawa, M. Tada-Umezaki, Y. Ota, and M. Nakahara. Quantum oracles in terms of universal gate set. *Int. J. Quant. Inf.*, 9:1363–1381, 2011.
- [48] D. Ventura and T. Martinez. Initializing the amplitude distribution of a quantum state. *Found. Phys. Lett.*, 12:547–559, 1999.
- [49] L. K. Grover. Quantum search on structured problems, 1998. arXiv: quant-ph/9802035.
- [50] D. E. Knuth. *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms, 3rd Ed.* Addison-Wesley, Reading, MA, 1997. Chap. 3.
- [51] M. Matsumoto and T. Nishimura. Mersenne Twister: a 623-dimensionally equidistributed uniform pseudorandom number generator. *ACM Trans. Model. Comput. Sim.*, 8:3–30, 1998. <http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/mt.html>.
- [52] A. Rukhin *et al.* A statistical test suite for random and pseudorandom number generators for cryptographic applications, 2010. NIST Special Publication 800-22, Revision 1a, <http://csrc.nist.gov/groups/ST/toolkit/rng/index.html>.